

# ON THE FAMILIES OF $q$ -EULER NUMBERS AND POLYNOMIALS AND THEIR APPLICATIONS

SERKAN ARACI, MEHMET ACIKGOZ, AND HASSAN JOLANY

**ABSTRACT.** In the present paper, we investigate special generalized  $q$ -Euler numbers and polynomials. Some earlier results of T. Kim in terms of  $q$ -Euler polynomials with weight  $\alpha$  can be deduced. For presentation of our formulas we apply the method of generating function and  $p$ -adic  $q$ -integral representation on  $\mathbb{Z}_p$ . We summarize our results as follows. In section 2, by using combinatorial techniques we present two formulas for  $q$ -Euler numbers with weight  $\alpha$ . In section 3, we derive distribution formula (Multiplication Theorem) for Dirichlet type of  $q$ -Euler numbers and polynomials with weight  $\alpha$ . Moreover we define partial Dirichlet type zeta function and Dirichlet  $q$ - $L$ -function, and obtain some interesting combinatorial identities for interpolating our new definitions. In addition, we derive behavior of the Dirichlet type of  $q$ -Euler  $L$ -function with weight  $\alpha$ ,  $\mathcal{L}_q^\chi(s, x | \alpha)$  at  $s = 0$ . Furthermore by using second kind stirling numbers, we obtain an explicit formula for Dirichlet type  $q$ -Euler numbers with weight  $\alpha$ , and  $\beta$ . Moreover a novel formula for  $q$ -Euler-Zeta function with weight  $\alpha$  in terms of nested series of  $\tilde{\zeta}_{E,q}(n | \alpha)$  is derived. In section 4, by introducing  $p$ -adic Dirichlet type of  $q$ -Euler measure with weight  $\alpha$ , and  $\beta$ , we obtain some combinatorial relations, which interpolate our previous results. In section 5, which is the main section of our paper. As an application, we introduce a novel concept of dynamics of the zeros of analytically continued  $q$ -Euler polynomials with weight  $\alpha$ .

## 1. Introduction

In this paper, we use notations like  $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{C}$ , where  $\mathbb{N}$  denotes the set of natural numbers,  $\mathbb{R}$  denotes the field of real numbers and  $\mathbb{C}$  also denotes the set of complex numbers. When one talks of  $q$ -extension,  $q$  is variously considered as an indeterminate, a complex number or a  $p$ -adic number.

Throughout this paper, we will assume that  $q \in \mathbb{C}$  with  $|q| < 1$ . The  $q$ -symbol  $[x : q]$  denotes as

$$[x : q] = \frac{q^x - 1}{q - 1}.$$

Originally,  $q$ -Euler numbers and polynomials were introduced by L. Carlitz in 1948 and gave properties of this polynomials (see [20], [21]). Recently, Taekyun Kim, by using  $p$ -adic  $q$ -integral in the  $p$ -adic integers ring, has added a weight to  $q$ -Bernoulli numbers and polynomials and gave surprising and fascinating identities of them (for details, see [8]). The  $q$ -Bernoulli numbers and polynomials with weight  $\alpha$  are related to weighted  $q$ -Bernstein polynomials which is shown by Kim (for details,

---

2000 *Mathematics Subject Classification.* Primary 05A10, 11B65; Secondary 11B68, 11B73.

*Key words and phrases.* Euler numbers and polynomials,  $q$ -Euler numbers and polynomials, weighted  $q$ -Euler numbers and polynomials, weighted  $q$ -Euler-Zeta function,  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$ .

see [7]). These polynomials have surprising properties in Analytic Numbers Theory and in  $p$ -adic analysis, especially, in Mathematical physics. Ryoo also constructed  $q$ -Euler numbers and polynomials with weight  $\alpha$  and introduced some properties of  $q$ -Euler numbers and polynomials with weight  $\alpha$  in "A note on the weighted  $q$ -Euler numbers and polynomials with weight  $\alpha$ , Advanced Studies Contemporary Mathematics 21 (2011), No. 1, 47-54."

Analytic continuation of  $q$ -Euler numbers and polynomials was investigated by Kim in [1]. In previous paper, Araci *et al.* also considered analytic continuation of weighted  $q$ -Genocchi numbers and polynomials and introduced some interesting ideas (for detail, see [26]). In this article, we also specify analytic continuation of weighted  $q$ -Euler numbers and polynomials. Also, we give some interesting identities by using generating function of Ryoo's weighted  $q$ -Euler polynomials.

Because in the literature of our present paper we use of  $p$ -adic Arithmetic and  $p$ -adic numbers, so we need to give short review on  $p$ -adic numbers. Historically the  $p$ -adic numbers were introduced by K. Hensel in 1908 in his book *Theorie der algebraischen Zahlen*, Leipzig, 1908 (for more informations on this subject, see [19]).

Let  $p$  be a prime number, fixed once and for all. If  $x$  is any rational number other than 0, we can write  $x$  in the form  $x = p^n \frac{a}{b}$ , where  $a, b \in \mathbb{Z}$  are relatively prime to  $p$  and  $n \in \mathbb{Z}$ . We now define

$$|x|_p = p^{-n} \text{ and } |0|_p = 0, \text{ and } \text{ord}_p(x) = n \text{ and } \text{ord}_p(0) = +\infty.$$

They satisfy the following properties,

$$\begin{aligned} &|x|_p \geq 0, \quad |x|_p = 0 \text{ if and only if } x = 0 \\ &|x + y|_p \leq \max\{|x|_p, |y|_p\} \text{ (the strong triangle inequality)} \\ &\quad \text{with} \\ &|x + y|_p = \max\{|x|_p, |y|_p\} \text{ if } |x|_p \neq |y|_p \text{ (the isosceles triangle principle)} \\ &|x \cdot y|_p = |x|_p \cdot |y|_p \end{aligned}$$

$|x|_p$  is called the  $p$ -adic valuation. Ostrowski proved that each nontrivial valuation on the field of rational numbers is equivalent either to the absolute value function or to some  $p$ -adic valuation. The completion of the field  $\mathbb{Q}$  of rational numbers with respect to the  $p$ -adic valuation  $|\cdot|_p$  is called the field of  $p$ -adic numbers and will be denoted  $\mathbb{Q}_p$ . The set

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$$

is the ring of  $p$ -adic integers. It can be easily proved that each  $p$ -adic number  $x$  can be written in the form

$$x = \sum_{n=-f}^{\infty} a_n p^n$$

where each  $a_n$  is one of the elements  $0, 1, \dots, p-1$ , and  $f \in \mathbb{Z}$ . This is called the Hensel representation of  $p$ -adic numbers. With this representation, one obtain for  $x \in \mathbb{Q}_p$ ,  $\text{ord}_p(x) = +\infty$  if  $a_i = 0$  for all  $i$  and  $\text{ord}_p(x) = \min\{s \mid a_s \neq 0\}$ , otherwise. Moreover we can write

$$|x|_p = p^{-\text{ord}_p(x)}.$$

## 2. Properties of the $q$ -Euler Numbers and polynomials with weight $\alpha$

For  $\alpha \in \mathbb{N} \cup \{0\}$ , the weighted  $q$ -Euler polynomials are given as:

For  $x \in \mathbb{C}$ ,

$$(2.1) \quad \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x | \alpha) \frac{t^n}{n!} = [2 : q] \sum_{n=0}^{\infty} (-1)^n q^n e^{t[n+x:q^\alpha]}.$$

As a special case, substituting  $x = 0$  into (2.1),  $\tilde{E}_{n,q}(0 | \alpha) := \tilde{E}_{n,q}(\alpha)$  are called weighted  $q$ -Euler numbers. By (2.1), we readily derive the following

$$(2.2) \quad \tilde{E}_{n,q}(x | \alpha) = \frac{[2 : q]}{[\alpha : q]^n (1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{1 + q^{\alpha l + 1}},$$

where  $\binom{n}{l}$  is the binomial coefficient. By expression (2.1), we see that

$$(2.3) \quad \tilde{E}_{n,q}(x | \alpha) = q^{-\alpha x} \left( q^{\alpha x} \tilde{E}_q(\alpha) + [x : q^\alpha] \right)^n,$$

with the usual convention of replacing  $\left( \tilde{E}_q(\alpha) \right)^n$  by  $\tilde{E}_{n,q}(\alpha)$  (for details, see [14]).

Let  $\tilde{H}_q^{(\alpha)}(x, t)$  be the generating function of weighted  $q$ -Euler polynomials as follows:

$$(2.4) \quad \tilde{H}_q^{(\alpha)}(x, t) = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x | \alpha) \frac{t^n}{n!}.$$

Then, we easily notice that

$$(2.5) \quad \tilde{H}_q^{(\alpha)}(x, t) = [2 : q] \sum_{n=0}^{\infty} (-1)^n q^n e^{t[n+x:q^\alpha]}.$$

From expressions (2.4) and (2.5), we procure the followings:

For  $k$  (=even) and  $n, \alpha \in \mathbb{N} \cup \{0\}$ , we have

$$(2.6) \quad \tilde{E}_{n,q}(\alpha) - q^k \tilde{E}_{n,q}(k | \alpha) = [2 : q] \sum_{l=0}^{k-1} (-1)^l q^l [l : q^\alpha]^n.$$

For  $k$  (=odd) and  $n, \alpha \in \mathbb{N} \cup \{0\}$ , we have

$$(2.7) \quad q^k \tilde{E}_{n,q}(k | \alpha) + \tilde{E}_{n,q}(\alpha) = [2 : q] \sum_{l=0}^{k-1} (-1)^l q^l [l : q^\alpha]^n.$$

Via Eq. (2.5), we easily obtain the following:

$$(2.8) \quad \tilde{E}_{n,q}(x | \alpha) = q^{-\alpha x} \sum_{k=0}^n \binom{n}{k} q^{\alpha k x} \tilde{E}_{k,q}(\alpha) [x : q^\alpha]^{n-k}.$$

From (2.6)-(2.8), we get the following theorem.

**Theorem 1.** *Let  $k$  be even positive integer. Then we have*

$$(2.9) \quad [2 : q] \sum_{l=0}^{k-1} (-1)^l q^l [l : q^\alpha]^n \\ = \left(1 - q^{k(1-\alpha+\alpha n)}\right) \tilde{E}_{n,q}(\alpha) - q^{k(1-\alpha)} \sum_{l=0}^{n-1} \binom{n}{l} q^{\alpha l k} \tilde{E}_{l,q}(\alpha) [k : q^\alpha]^{n-l}.$$

**Theorem 2.** *Let  $k$  be an odd positive integer. Then, we procure the following*

$$(2.10) \quad [2 : q] \sum_{l=0}^{k-1} (-1)^l q^l [l : q^\alpha]^n \\ = \left(q^{k(1-\alpha+\alpha n)} + 1\right) \tilde{E}_{n,q}(\alpha) + q^{k(1-\alpha)} \sum_{l=0}^{n-1} \binom{n}{l} q^{\alpha l k} \tilde{E}_{l,q}(\alpha) [k : q^\alpha]^{n-l}.$$

### 3. $q$ -Euler-Zeta function with weight $\alpha$

The familiar Euler polynomials are defined by

$$(3.1) \quad \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi \text{ cf. [4].}$$

For  $s \in \mathbb{C}$ ,  $x \in \mathbb{R}$  with  $0 \leq x < 1$ , Euler-Zeta function is given by

$$(3.2) \quad \zeta_E(s, x) = 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+x)^s},$$

and

$$(3.3) \quad \zeta_E(s) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m^s}.$$

By expressions (3.1), (3.2) and (3.3), Euler-Zeta functions are related to the Euler numbers as follows:

$$\zeta_E(-n) = E_n.$$

Moreover, it is simple to see

$$\zeta_E(-n, x) = E_n(x).$$

The weighted  $q$ -Euler Hurwitz-Zeta type function is defined by

$$\tilde{\zeta}_{E,q}(s, x | \alpha) = [2 : q] \sum_{m=0}^{\infty} \frac{(-1)^m q^m}{[m+x : q^\alpha]^s}.$$

Similarly, weighted  $q$ -Euler-Zeta function is given by

$$\tilde{\zeta}_{E,q}(s | \alpha) = [2 : q] \sum_{m=1}^{\infty} \frac{(-1)^m q^m}{[m : q^\alpha]^s}.$$

For  $n, \alpha \in \mathbb{N} \cup \{0\}$ , we have

$$\tilde{\zeta}_{E,q}(-n | \alpha) = \tilde{E}_{n,q}(\alpha) \text{ (see [14]).}$$

We now consider the function  $\tilde{E}_q(n : \alpha)$  as the analytic continuation of weighted  $q$ -Euler numbers. All the weighted  $q$ -Euler numbers agree with  $\tilde{E}_q(n : \alpha)$ , the

analytic continuation of weighted  $q$ -Euler numbers evaluated at  $n$ . For  $n \geq 0$ ,  $\tilde{E}_q(n : \alpha) = \tilde{E}_{n,q}(\alpha)$ .

We can now state  $\tilde{E}_q(s : \alpha)$  in terms of  $\tilde{\zeta}_{E,q}(s | \alpha)$ , the derivative of  $\tilde{\zeta}_{E,q}(s : \alpha)$

$$\tilde{E}_q(s : \alpha) = \tilde{\zeta}_{E,q}(-s | \alpha), \quad \tilde{E}_q^\vee(s : \alpha) = \tilde{\zeta}_{E,q}^\vee(-s | \alpha).$$

For  $n, \alpha \in \mathbb{N} \cup \{0\}$

$$\tilde{E}_q(2n : \alpha) = \tilde{\zeta}_{E,q}(-2n | \alpha).$$

This is suitable for the differential of the functional equation and so supports the coherence of  $\tilde{E}_q(s : \alpha)$  and  $\tilde{E}_q^\vee(s : \alpha)$  with  $\tilde{E}_{n,q}(\alpha)$  and  $\tilde{\zeta}_{E,q}(s | \alpha)$ . From the analytic continuation of weighted  $q$ -Euler numbers, we derive as follows:

$$\tilde{E}_q(s : \alpha) = \tilde{\zeta}_{E,q}(-s | \alpha) \quad \text{and} \quad \tilde{E}_q(-s : \alpha) = \tilde{\zeta}_{E,q}(s | \alpha).$$

Moreover, we derive the following:

For  $n \in \mathbb{N}$

$$\tilde{E}_{-n,q}(\alpha) = \tilde{E}_q(-n : \alpha) = \tilde{\zeta}_{E,q}(n | \alpha).$$

The curve  $\tilde{E}_q(s : a)$  runs through the points  $\tilde{E}_{-s,q}(\alpha)$  and grows  $\sim n$  asymptotically  $(-n) \rightarrow -\infty$ . The curve  $\tilde{E}_q(s : a)$  runs through the point  $\tilde{E}_q(-n : a)$ . Then, we procure the following:

$$\begin{aligned} \lim_{n \rightarrow \infty} \tilde{E}_q(-n : \alpha) &= \lim_{n \rightarrow \infty} \tilde{\zeta}_{E,q}(n | \alpha) = \lim_{n \rightarrow \infty} \left( [2 : q] \sum_{m=1}^{\infty} \frac{(-1)^m q^m}{[m : q^\alpha]^n} \right) \\ &= \lim_{n \rightarrow \infty} \left( -q[2 : q] + [2 : q] \sum_{m=2}^{\infty} \frac{(-1)^m q^m}{[m : q^\alpha]^n} \right) = -q^2 [2 : q^{-1}]. \end{aligned}$$

From this, we note that

$$\tilde{E}_q(-n : \alpha) = \tilde{\zeta}_{E,q}(n | \alpha) \mapsto \tilde{E}_q(-s : \alpha) = \tilde{\zeta}_{E,q}(s | \alpha).$$

NOTATIONS: Assume that  $p$  be a fixed odd prime number. Throughout this paper we use the following notations. By  $\mathbb{Z}_p$  we denote the ring of  $p$ -adic rational integers,  $\mathbb{Q}$  denotes the field of rational numbers,  $\mathbb{Q}_p$  denotes the field of  $p$ -adic rational numbers, and  $\mathbb{C}_p$  denotes the completion of algebraic closure of  $\mathbb{Q}_p$ . Let  $\mathbb{N}$  be the set of natural numbers and  $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ . The  $p$ -adic absolute value is defined by  $|p|_p = \frac{1}{p}$ . In this paper we assume  $|q-1|_p < 1$  as an indeterminate. Let  $UD(\mathbb{Z}_p)$  be the space of uniformly differentiable functions on  $\mathbb{Z}_p$ . For a positive integer  $d$  with  $(d, p) = 1$ , set

$$\begin{aligned} X &= X_d = \varprojlim_n \mathbb{Z}/dp^n \mathbb{Z}, \\ X^* &= \bigcup_{\substack{0 < a < dp^n \\ (a, p) = 1}} a + dp^n \mathbb{Z}_p \end{aligned}$$

and

$$a + dp^n \mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^n}\},$$

where  $a \in \mathbb{Z}$  satisfies the condition  $0 \leq a < dp^n$ .

Firstly, for introducing fermionic  $p$ -adic  $q$ -integral, we need some basic information which we state here. A measure on  $\mathbb{Z}_p$  with values in a  $p$ -adic Banach space

$B$  is a continuous linear map

$$f \mapsto \int f(x) \mu = \int_{\mathbb{Z}_p} f(x) \mu(x)$$

from  $C^0(\mathbb{Z}_p, \mathbb{C}_p)$ , (continuous function on  $\mathbb{Z}_p$ ) to  $B$ . We know that the set of locally constant functions from  $\mathbb{Z}_p$  to  $\mathbb{Q}_p$  is dense in  $C^0(\mathbb{Z}_p, \mathbb{C}_p)$  so.

Explicitly, for all  $f \in C^0(\mathbb{Z}_p, \mathbb{C}_p)$ , the locally constant functions

$$f_n = \sum_{i=0}^{p^n-1} f(i) 1_{i+p^n\mathbb{Z}_p} \rightarrow f \text{ in } C^0$$

Now, set  $\mu(i + p^n\mathbb{Z}_p) = \int_{\mathbb{Z}_p} 1_{i+p^n\mathbb{Z}_p} \mu$ . Then  $\int_{\mathbb{Z}_p} f \mu$ , is given by the following Riemannian sum

$$\int_{\mathbb{Z}_p} f \mu = \lim_{n \rightarrow \infty} \sum_{i=0}^{p^n-1} f(i) \mu(i + p^n\mathbb{Z}_p)$$

T. Kim introduced  $\mu$  as follows:

$$\mu_{-q}(a + p^n\mathbb{Z}_p) = \frac{(-q)^a}{[p^n]_{-q}}$$

So, for  $f \in UD(\mathbb{Z}_p)$ , the fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  is defined by Kim as follows:

$$\begin{aligned} (3.4) \quad I_{-q}(f) &= \int_{\mathbb{Z}_p} f(\eta) d\mu_{-q}(\eta) \\ &= \lim_{n \rightarrow \infty} \frac{1}{[p^n]_{-q}} \sum_{\eta=0}^{p^n-1} q^\eta f(\eta) (-1)^\eta. \end{aligned}$$

Let  $\chi$  be the Dirichlet's character with conductor  $d (= \text{odd}) \in \mathbb{N}$  and let us take  $f(\eta) = \chi(\eta) [x + \eta : q^\alpha]^n$ , then we define Dirichlet's type of  $q$ -Euler numbers and polynomials with weight  $\alpha$  as follows:

$$(3.5) \quad \tilde{\mathcal{E}}_{n,q}^\chi(x | \alpha) = \int_{\mathbb{Z}_p} \chi(\eta) [x + \eta : q^\alpha]^n d\mu_{-q}(\eta).$$

From (3.4), we have the following well-known equality.

$$(3.6) \quad q^d \int_{\mathbb{Z}_p} f(\eta + d) d\mu_{-q}(\eta) + (-1)^{d-1} \int_{\mathbb{Z}_p} f(\eta) d\mu_{-q}(\eta) = [2 : q] \sum_{l=0}^{d-1} q^l (-1)^{d-1-l} f(l).$$

By expressions of (3.5) and (3.6), for  $d (= \text{odd})$  positive integer, we have the following

$$(3.7) \quad \tilde{\mathcal{E}}_{n,q}^\chi(x | \alpha) = \frac{[2 : q]}{[\alpha : q]^n (1 - q)^n} \sum_{j=0}^n \binom{n}{j} (-1)^j q^{\alpha j x} \sum_{l=0}^{d-1} \chi(l) q^l (-1)^l \frac{q^{\alpha j l}}{q^{(\alpha j + 1)d} + 1}.$$

Substituting  $x = 0$  in (3.7),  $\tilde{\mathcal{E}}_{n,q}^\chi(0 \mid \alpha) := \tilde{\mathcal{E}}_{n,q}^\chi(\alpha)$  are called Dirichlet type of  $q$ -Euler numbers with weight  $\alpha$ . That is, we easily derive the following

$$(3.8) \quad \tilde{\mathcal{E}}_{n,q}^\chi(\alpha) = \frac{[2 : q]}{(1 - q^\alpha)^n} \sum_{j=0}^n \binom{n}{j} (-1)^j \sum_{l=0}^{d-1} \chi(l) (-1)^l \frac{q^{(\alpha j+1)l}}{q^{(\alpha j+1)d} + 1}.$$

**Theorem 3.** *Let  $\chi$  be Dirichlet's character and for any  $n \in \mathbb{N}^*$ . Then we have*

$$\tilde{\mathcal{E}}_{n,q}^\chi(x \mid \alpha) = \sum_{k=0}^n \binom{n}{k} q^{\alpha k x} \tilde{\mathcal{E}}_{k,q}^\chi(\alpha) [x : q^\alpha]^{n-k}.$$

*Proof.* By using (3.5) and (3.8), becomes

$$\begin{aligned} \tilde{\mathcal{E}}_{n,q}^\chi(x \mid \alpha) &= \int_{\mathbb{Z}_p} \chi(\eta) [x + \eta : q^\alpha]^n d\mu_{-q}(\eta) \\ &= \int_{\mathbb{Z}_p} \chi(\eta) ([x : q^\alpha] + q^{\alpha x} [\eta : q^\alpha])^n d\mu_{-q}(\eta). \end{aligned}$$

From this, by using binomial theorem, we can write the following

$$\begin{aligned} &\sum_{k=0}^n \binom{n}{k} q^{\alpha k x} [x : q^\alpha]^{n-k} \int_{\mathbb{Z}_p} \chi(\eta) [\eta : q^\alpha]^k d\mu_{-q}(\eta) \\ &= \sum_{k=0}^n \binom{n}{k} q^{\alpha k x} [x : q^\alpha]^{n-k} \tilde{\mathcal{E}}_{k,q}^\chi(\alpha). \end{aligned}$$

Thus, we complete the proof of the theorem.  $\square$

**Theorem 4.** *The following identity*

$$\tilde{\mathcal{E}}_{n,q}^\chi(dx \mid \alpha) = \frac{[d : q^\alpha]}{[d : -q]} \sum_{a=0}^{d-1} (-1)^a \chi(a) q^a \tilde{E}_{n,q^d} \left( x + \frac{a}{d} \mid \alpha \right)$$

*holds true.*

*Proof.* To prove this, we compute as follows:

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{1}{[dp^n : -q]} \sum_{y=0}^{dp^n-1} (-q)^y \chi(y) [x + y : q^\alpha]^n \\ &= \frac{1}{[d : -q]} \lim_{n \rightarrow \infty} \frac{1}{[p^n : -q^d]} \sum_{y=0}^{p^n-1} \sum_{a=0}^{d-1} (-q)^{a+dy} \chi(a+dy) [x + a + dy : q^\alpha]^n \\ &= \frac{[d : q^\alpha]}{[d : -q]} \sum_{a=0}^{d-1} (-q)^a \chi(a) \lim_{n \rightarrow \infty} \frac{1}{[p^n]_{-q^d}} \sum_{y=0}^{p^n-1} (-q)^{dy} \left[ \frac{x+a}{d} + y : q^{d\alpha} \right]^n \\ &= \frac{[d : q^\alpha]}{[d : -q]} \sum_{a=0}^{d-1} (-q)^a \chi(a) \tilde{E}_{n,q^d} \left( \frac{x+a}{d} \mid \alpha \right). \end{aligned}$$

So, we get the desired result and proof is complete.  $\square$

By (3.7), we procure the following:

$$(3.9) \quad \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_{n,q}^\chi(x \mid \alpha) \frac{t^n}{n!} = [2 : q] \sum_{m=0}^{\infty} q^m \chi(m) (-1)^m e^{t[x+m:q^\alpha]}.$$

By applying derivative operator of order  $k$  as  $\frac{d^k}{dt^k} |_{t=0}$ , we have the following

$$\tilde{\mathcal{E}}_{k,q}^\chi(x | \alpha) = [2 : q] \sum_{m=0}^{\infty} q^m \chi(m) (-1)^m [x + m : q^\alpha]^k.$$

That is, we can define Dirichlet  $q$ - $L$ -function as follows:

$$(3.10) \quad \mathcal{L}_q^\chi(s, x | \alpha) = [2 : q] \sum_{m=0}^{\infty} \frac{q^m \chi(m) (-1)^m}{[x + m : q^\alpha]^s}.$$

**Lemma 1.** *The following equality holds true:*

$$\mathcal{L}_q^\chi(-k, x | \alpha) = \tilde{\mathcal{E}}_{k,q}^\chi(x | \alpha).$$

*Proof.* Substituting  $s = -k$  into (3.10), we arrive at the desired result.  $\square$

Now also, we define partial Dirichlet type zeta function as follows:

$$(3.11) \quad \mathcal{H}_q^\chi(s : x : a : F | \alpha) = [2 : q] \sum_{m \equiv a \pmod{F}}^{\infty} \frac{q^m \chi(m) (-1)^m}{[x + m : q^\alpha]^s}.$$

Now, for interpolating partial Dirichlet type zeta function, we rewrite it in terms of weighted  $q$ -Euler Hurwitz-Zeta function as follows.

**Theorem 5.** *For  $F \equiv 1 \pmod{2}$ , then the following equality holds true:*

$$(3.12) \quad \mathcal{H}_q^\chi(s : x : a : F | \alpha) = \frac{[2 : q] q^a (-1)^a \chi(a)}{[F : q^\alpha]^s} \tilde{\zeta}_{E,q^F} \left( s, \frac{x+a}{F} | \alpha \right).$$

*Proof.* By expression of (3.11), we compute as follows:

$$\begin{aligned} \mathcal{H}_q^\chi(s : x : a : F | \alpha) &= [2 : q] \sum_{m \equiv a \pmod{F}}^{\infty} \frac{q^m \chi(m) (-1)^m}{[x + m : q^\alpha]^s} \\ &= [2 : q] \sum_{m=0}^{\infty} \frac{q^{mF+a} \chi(mF+a) (-1)^{mF+a}}{[x + mF + a : q^\alpha]^s} \\ &= \frac{[2 : q] q^a (-1)^a \chi(a)}{[F : q^\alpha]^s} \sum_{m=0}^{\infty} \frac{(q^F)^m (-1)^m}{\left[\frac{x+a}{F} + m : q^{F\alpha}\right]^s}. \end{aligned}$$

Thus, we arrive at the desired result.  $\square$

If we put  $s = -n$  into (3.12), then, we can write partial Dirichlet type Zeta function in terms of weighted  $q$ -Euler numbers

$$(3.13) \quad \mathcal{H}_q^\chi(-n : x : a : F | \alpha) = [2 : q] q^a (-1)^a \chi(a) [F : q^\alpha]^n \tilde{E}_{n,q^F} \left( \frac{x+a}{F} | \alpha \right).$$

**Theorem 6.** *The following identity*

$$(3.14) \quad \mathcal{H}_q^\chi(s : x : a : F | \alpha) = \frac{[2 : q] q^a (-1)^a \chi(a)}{[x + a : q^\alpha]^s} \sum_{k=0}^{\infty} q^{\alpha k(x+a)} \binom{-s}{k} \left( \frac{[F : q^\alpha]}{[x + a : q^\alpha]} \right)^k \tilde{E}_{k,q^F}$$

*holds true.*

*Proof.* Taking  $n = -s$  into (3.13) and some manipulation by using combinatorial techniques, we can reach to the proof of the theorem.  $\square$



If we substitute  $s = -n$  into (3.14), then, (3.14) reduces to (3.13). Now also, we give the following theorem.

**Theorem 7.** *Let  $d \equiv 1 \pmod{2}$ , then, we have*

$$\mathcal{L}_q^\chi(s, x \mid \alpha) = \frac{[2 : q]}{[2 : q^d][d : q^\alpha]^s} \sum_{l=0}^{d-1} (-1)^l \chi(l) q^l \tilde{\zeta}_{E, q^d} \left( s, \frac{x+l}{d} \mid \alpha \right).$$

*Proof.* By using (3.10), we compute as follows:

$$\begin{aligned} \mathcal{L}_q^\chi(s, x \mid \alpha) &= [2 : q] \sum_{m=0}^{\infty} \sum_{l=0}^{d-1} \frac{q^{l+md} \chi(l+md) (-1)^{l+md}}{[x+l+md : q^\alpha]^s} \\ &= \frac{[2 : q]}{[2 : q^d][d : q^\alpha]^s} \sum_{l=0}^{d-1} (-1)^l \chi(l) q^l \tilde{\zeta}_{E, q^d} \left( s, \frac{x+l}{d} \mid \alpha \right). \end{aligned}$$

Thus, we prove the above theorem.  $\square$

By means of the above theorem and (3.12), we have the following corollary.

**Corollary 1.** *The following equality*

$$(3.15) \quad \mathcal{L}_q^\chi(s, x \mid \alpha) = \frac{1}{[2 : q^d]} \sum_{l=0}^{d-1} \mathcal{H}_q^\chi(s : x : l : d \mid \alpha)$$

*holds true.*

By (3.14) and (3.15), we have the following corollary.

**Corollary 2.** *The following nice identity*

$$\mathcal{L}_q^\chi(s, x \mid \alpha) = \frac{[2 : q]}{[2 : q^d]} \sum_{l=0}^{d-1} \frac{(-1)^l \chi(l)}{[x+l : q^\alpha]^s} \sum_{k=0}^{\infty} q^{\alpha k(x+l)+l} \binom{-s}{k} \tilde{E}_{k, q^F} \left( \frac{[F : q^\alpha]}{[x+l : q^\alpha]} \right)^k$$

*holds true.*

By using (3.13) and (3.15), we derive behavior of the Dirichlet type of  $q$ -Euler  $L$ -function with weight  $\alpha$  at  $s = 0$  as follows:

**Theorem 8.** *The following identity holds true:*

$$\mathcal{L}_q^\chi(0, x \mid \alpha) = \frac{[2 : q]}{[2 : q^d]} \sum_{l=0}^{d-1} (-1)^l \chi(l) q^l.$$

Now also, we define Dirichlet type  $q$ -Euler polynomials with weight  $\alpha$  and  $\beta$  with the following expression

$$(3.16) \quad \tilde{\mathcal{E}}_{n, q}^\chi(x \mid \alpha : \beta) = \int_{\mathbb{Z}_p} [x + \eta : q^\alpha]^n \chi(\eta) d\mu_{-q^\beta}(\eta).$$

Taking  $x = 0$  into (3.16), we have  $\tilde{\mathcal{E}}_{n, q}^\chi(0 \mid \alpha : \beta) := \tilde{\mathcal{E}}_{n, q}^\chi(\alpha : \beta)$  which is called Dirichlet type  $q$ -Euler numbers. Then, by (3.16), we easily derive the following

$$(3.17) \quad \tilde{\mathcal{E}}_{n, q}^\chi(x \mid \alpha : \beta) = \sum_{l=0}^n \binom{n}{l} q^{\alpha l x} \tilde{\mathcal{E}}_{l, q}^\chi(\alpha : \beta) [x : q^\alpha]^{n-l}.$$

**Theorem 9.** *The following equality holds true:*

$$\begin{aligned} \tilde{\mathcal{E}}_{n,q}^{\chi}(x | \alpha : \beta) &= \sum_{l=0}^n \binom{n}{l} \tilde{\mathcal{E}}_{l,q}^{\chi}(\alpha : \beta) \sum_{j=0}^l \binom{l}{j} (q^{\alpha} - 1)^j (n - l + j)! (-1)^{n-l+j} \\ &\quad \times \sum_{m,n=0}^{\infty} \binom{n-l+j+m-1}{m} \alpha^n q^{\alpha m} (\log q)^n \mathcal{S}(n, n-l+j) \frac{x^n}{n!}. \end{aligned}$$

*Proof.* To prove this, by applying (3.17), we easily discover the following assertion

$$\tilde{\mathcal{E}}_{n,q}^{\chi}(x | \alpha : \beta) = \sum_{l=0}^n \binom{n}{l} \tilde{\mathcal{E}}_{l,q}^{\chi}(\alpha : \beta) \sum_{j=0}^l \binom{l}{j} (q^{\alpha} - 1)^j [x : q^{\alpha}]^{n-l+j}.$$

The Second kind Stirling numbers are defined by means of the following generating function.

$$(3.18) \quad \sum_{n=0}^{\infty} \mathcal{S}(n, k) \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}$$

(for details on this subject, see [12]).  $t$  replace by  $\alpha x \log q$  in (3.18), then, we easily derive the following

$$(3.19) \quad [x : q^{\alpha}]^k = k! (-1)^k \sum_{m,n=0}^{\infty} \binom{k+m-1}{m} \alpha^n q^{\alpha m} (\log q)^n \mathcal{S}(n, k) \frac{x^n}{n!},$$

where  $\sum_{m,n=0}^{\infty} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}$ . Thus, by (3.18) and (3.19), we get the desired result and proof is complete.  $\square$

**Theorem 10.** *The following equality*

$$\tilde{\mathcal{E}}_{n,q}^{\chi}(x | \alpha : \beta) = \frac{[d : q^{\alpha}]^n}{[d : -q^{\beta}]} \sum_{a=0}^{d-1} (-q)^a \chi(a) \tilde{E}_{n,q^d} \left( \frac{x+a}{d} | \alpha : \beta \right)$$

*holds true.*

*Proof.* By applying the  $p$ -adic integral representation on the Dirichlet type of  $q$ -Euler polynomials with weight  $\alpha$  and  $\beta$ , we compute as follows:

$$\begin{aligned} \tilde{\mathcal{E}}_{n,q}^{\chi}(x | \alpha : \beta) &= \int_{\mathbb{Z}_p} \chi(\eta) [x + \eta : q^{\alpha}]^n d\mu_{-q^{\beta}}(\eta) \\ &= \lim_{n \rightarrow \infty} \frac{1}{[dp^n : -q^{\beta}]} \sum_{y=0}^{dp^n-1} (-q)^y \chi(y) [x + y : q^{\alpha}] \\ &= \frac{1}{[d : -q^{\beta}]} \lim_{n \rightarrow \infty} \frac{1}{[p^n : -q^{d\beta}]} \sum_{y=0}^{p^n-1} \sum_{a=0}^{d-1} (-q)^{a+dy} \chi(a+dy) [x + a + dy : q^{\alpha}]^n \\ &= \frac{[d : q^{\alpha}]^n}{[d : -q^{\beta}]} \sum_{a=0}^{d-1} (-q)^a \chi(a) \lim_{n \rightarrow \infty} \frac{1}{[p^n : -q^d]} \sum_{y=0}^{p^n-1} (-q^{d\beta})^y \left[ \frac{x+a}{d} + y : q^{d\alpha} \right]^n \\ &= \frac{[d : q^{\alpha}]^n}{[d : -q^{\beta}]} \sum_{a=0}^{d-1} (-q)^a \chi(a) \tilde{E}_{n,q^d} \left( \frac{x+a}{d} | \alpha : \beta \right). \end{aligned}$$

Here,  $\tilde{E}_{n,q^d} \left( \frac{x+a}{d} \mid \alpha : \beta \right)$  is defined by Ryoo in [16], which is called  $q$ -Euler polynomials with weight  $(\alpha, \beta)$ . As a result, we have the proof of the theorem.  $\square$

#### 4. On $p$ -adic Dirichlet type of $q$ -Euler measure with weight $\alpha$ and $\beta$

Now, we introduce a map  $\mu_{k,q}^{(\alpha,\beta)}(a + p^n \mathbb{Z}_p)$  on the balls in  $\mathbb{Z}_p$  as follows:

$$(4.1) \quad \mu_{k,q}^{(\alpha,\beta)}(a + p^n \mathbb{Z}_p \mid \chi) = \frac{[p^n : q^\alpha]^k}{[p^n : -q^\beta]} \chi(a) (-1)^a q^a f_{k,p^n} \left( \frac{\{a\}_n}{p^n} \mid \alpha : \beta \right)$$

where  $\{a\}_n \equiv a \pmod{p^n}$ .

**Theorem 11.** *Let  $\alpha, k \in \mathbb{N}$ . Then we specify that  $\mu_{k,q}^{(\alpha,\beta)}$  is  $p$ -adic measure on  $\mathbb{Z}_p$  if and only if*

$$f_{k,q^{p^n}} \left( \frac{a}{p^n} \mid \alpha : \beta \right) = \frac{[p^n : q^{p\alpha}]^k}{[p^n : -q^{p\beta}]} \sum_{b=0}^{p-1} (-1)^b q^{bp^n} f_{k,(q^{p^n})^p} \left( \frac{\frac{a}{p^n} + b}{p} \mid \alpha : \beta \right).$$

*Proof.* By similar method in [23], we can state the proof of this theorem. Therefore, we omit it.  $\square$

We now set as follows:

$$(4.2) \quad f_{k,q^{p^n}} \left( \frac{a}{p^n} \mid \alpha : \beta \right) = \tilde{E}_{n,q^{p^n}} \left( \frac{a}{p^n} \mid \alpha : \beta \right).$$

From (4.1) and (4.2), we easily see

$$(4.3) \quad \mu_{k,q}^{(\alpha,\beta)}(a + p^n \mathbb{Z}_p \mid \chi) = \frac{[p^n : q^\alpha]^k}{[p^n : -q^\beta]} \chi(a) (-1)^a q^a \tilde{E}_{k,q^{p^n}} \left( \frac{a}{p^n} \mid \alpha : \beta \right).$$

By (2.1) and (4.3), then, we have the following theorem.

**Theorem 12.** *For  $\alpha, k \in \mathbb{N}$ , we have*

$$\int_X d\mu_{k,q}^{(\alpha,\beta)}(x \mid \chi) = \tilde{\mathcal{E}}_{k,q}^\chi(\alpha : \beta).$$

*Proof.* By using combinatorial techniques, we compute as follows:

$$\begin{aligned} \int_X d\mu_{k,q}^{(\alpha,\beta)}(x \mid \chi) &= \lim_{n \rightarrow \infty} \frac{[dp^n : q^\alpha]^k}{[dp^n : -q^\beta]} \sum_{x=0}^{dp^n-1} \chi(x) (-1)^x q^x \tilde{E}_{k,q^{dp^n}} \left( \frac{x}{dp^n} \mid \alpha : \beta \right) \\ &= \frac{[d : q^\alpha]^k}{[d : -q^\beta]} \sum_{a=0}^{d-1} (-1)^a q^a \chi(a) \lim_{n \rightarrow \infty} \frac{[p^n : q^{d\alpha}]^k}{[p^n : -q^{d\beta}]} \sum_{x=0}^{p^n-1} (-1)^x q^{dx} \tilde{E}_{k,(q^d)^{p^n}} \left( \frac{\frac{a}{d} + x}{p^n} \mid \alpha : \beta \right) \\ &= \frac{[d : q^\alpha]^k}{[d : -q^\beta]} \sum_{a=0}^{d-1} (-1)^a q^a \chi(a) \tilde{E}_{n,q^d} \left( \frac{a}{d} \mid \alpha : \beta \right) \end{aligned}$$

so, we obtain the desired result.  $\square$

**Theorem 13.** *For any  $k \in \mathbb{N}$ , we get*

$$\int_{pX} d\mu_{k,q}^{(\alpha,\beta)}(x \mid \chi) = \chi(p) \frac{[p : q^\alpha]}{[p : -q^\beta]} \tilde{\mathcal{E}}_{k,q^p}^\chi(\alpha : \beta).$$

*Proof.* From (2.1) and (4.3), we derive the followings assertions

$$\begin{aligned}
\int_{pX} d\mu_{k,q}^{(\alpha,\beta)}(x | \chi) &= \lim_{n \rightarrow \infty} \frac{[dp^{n+1} : q^\alpha]^k}{[dp^{n+1} : -q^\beta]} \sum_{x=0}^{dp^n-1} \chi(px) (-1)^{px} q^{px} \tilde{E}_{k,q^{dp^n}} \left( \frac{px}{dp^{n+1}} | \alpha : \beta \right) \\
&= \chi(p) \frac{[p : q^\alpha]}{[p : -q^\beta]} \frac{[d : q^{p\alpha}]^k}{[d : -q^{p\beta}]} \sum_{a=0}^{d-1} \left\{ \begin{aligned} &(-1)^a q^{pa} \chi(a) \lim_{n \rightarrow \infty} \frac{[p^n : q^{dp\alpha}]^k}{[p^n : -q^{dp\beta}]} \\ &\times \sum_{x=0}^{p^n-1} (-1)^x q^{pdx} \tilde{E}_{k,(q^d)^{p^n}} \left( \frac{dp(\frac{a}{d} + x)}{pdp^n} | \alpha : \beta \right) \end{aligned} \right\} \\
&= \chi(p) \frac{[p : q^\alpha]}{[p : -q^\beta]} \frac{[d : q^{p\alpha}]^k}{[d : -q^{p\beta}]} \sum_{a=0}^{d-1} (-1)^a q^{pa} \chi(a) \tilde{E}_{k,q^{pd}} \left( \frac{a}{d} | \alpha : \beta \right) \\
&= \chi(p) \frac{[p : q^\alpha]}{[p : -q^\beta]} \tilde{\mathcal{E}}_{k,q^p}^\chi(\alpha : \beta).
\end{aligned}$$

Thus, we get the desired result and proof is complete.  $\square$

By the same method which we used in above theorem, by a little bit manipulations we can state the following theorem.

**Theorem 14.** For  $c(\neq 1) \in X^*$ , we have

$$\int_{pX} d\mu_{k,q^{\frac{1}{c}}}^{(\alpha,\beta)}(cx | \chi) = \chi\left(\frac{p}{c}\right) \frac{[p : q^{\frac{\alpha}{c}}]}{[p : (-q^\beta)^{\frac{1}{c}}]} \tilde{\mathcal{E}}_{k,q^{\frac{p}{c}}}^\chi(\alpha : \beta).$$

**Theorem 15.** For  $c(\neq 1) \in X^*$ , we have

$$\int_X d\mu_{k,q^{\frac{1}{c}}}^{(\alpha,\beta)}(cx | \chi) = \chi\left(\frac{1}{c}\right) \tilde{\mathcal{E}}_{k,q^{\frac{1}{c}}}^\chi(\alpha : \beta).$$

We can define the following identity:

$$\mu_{k,c,q}^{(\alpha,\beta)}(U | \chi) = \mu_{k,q}^{(\alpha,\beta)}(U | \chi) - c^{-1} \frac{[c^{-1} : q^\alpha]^k}{[c^{-1} : -q^\beta]} \mu_{k,q^{\frac{1}{c}}}^{(\alpha,\beta)}(cU | \chi)$$

here  $U$  is any compact open subset of  $\mathbb{Z}_p$ , it can be written as a finite disjoint union of sets

$$U = \bigcup_{j=1}^k (a_j + p^n \mathbb{Z}_p),$$

where  $n \in \mathbb{N}$  and  $a_1, a_2, \dots, a_k \in \mathbb{Z}$  with  $0 \leq a_i < p^n$  for  $i = 1, 2, \dots, k$ .

**Theorem 16.** For  $c(\neq 1) \in X^*$ , we procure the following

$$\int_{X^*} d\mu_{k,c,q}^{(\alpha,\beta)}(cx | \chi) = (1 - \chi^p) \left(1 - c^{-1} \chi^{c^{-1}}\right) \tilde{\mathcal{E}}_{k,q}^\chi(\alpha : \beta)$$

where the operator  $\chi^y := \chi^{y,k,\alpha;q}$  on  $f(q)$  is defined by

$$\chi^y f(q) = \chi^{y,k,\alpha;q} f(q) = \frac{[y : q^\alpha]}{[y : -q^\beta]} \chi(y) f(q^y)$$

That is, we can write

$$\chi^{x,k,\alpha;q} \circ \chi^{y,k,\alpha;q} f(q) = \chi^{xy,k,\alpha;q} f(q) = \chi^{xy} f(q).$$

*Proof.* To prove this, we assume that  $f(q) = \tilde{\mathcal{E}}_{k,q}^\chi(\alpha : \beta)$ . Then, we get

$$\left\{ \begin{array}{l} \tilde{\mathcal{E}}_{k,q}^\chi(\alpha : \beta) - \chi(p) \frac{[p;q^\alpha]}{[p;-q^\beta]} \tilde{\mathcal{E}}_{k,q^p}^\chi(\alpha : \beta) - c^{-1} \frac{[c^{-1};q^\alpha]^k}{[c^{-1};-q^\beta]} \\ \times \chi\left(\frac{1}{c}\right) \tilde{\mathcal{E}}_{k,q^{\frac{1}{c}}}^\chi(\alpha : \beta) + \chi\left(\frac{p}{c}\right) \frac{[p;q^{\frac{\alpha}{c}}]}{[p;(-q^\beta)^{\frac{1}{c}}]} \tilde{\mathcal{E}}_{k,q^{\frac{p}{c}}}^\chi(\alpha : \beta) \end{array} \right\}$$

From this, we derive the following

$$(1 - \chi^p) \left(1 - c^{-1} \chi^{c^{-1}}\right) \tilde{\mathcal{E}}_{k,q}^\chi(\alpha : \beta)$$

Then, we complete the proof of theorem.  $\square$

### 5. Analytic continuation of $q$ -Euler Polynomials with weight $\alpha$

The concept of analytic continuation just means enlarging the domain without giving up the property of being differentiable, i.e. holomorphic or meromorphic. More precisely Let  $f_1$  and  $f_2$  be analytic functions on domains  $\Omega_1$  and  $\Omega_2$ , respectively, and suppose that the intersection  $\Omega_1 \cap \Omega_2$  is not empty and that  $f_1 = f_2$  on  $\Omega_1 \cap \Omega_2$ . Then  $f_2$  is called an analytic continuation of  $f_1$  to  $\Omega_2$ , and vice versa. Moreover, if it exists, the analytic continuation of  $f_1$  to  $\Omega_2$  is unique. By means of analytic continuation, starting from a representation of a function by any one power series, any number of other power series can be found which together define the value of the function at all points of the domain. Furthermore, any point can be reached from a point without passing through a singularity of the function, and the aggregate of all the power series thus obtained constitutes the analytic expression of the function. So we are ready to state analytic continuation of  $q$ -Euler polynomials with weight  $\alpha$  as follows.

For coherence with the redefinition of  $\tilde{E}_{n,q}(\alpha) = \tilde{E}_q(n : \alpha)$ , we have

$$\tilde{E}_{n,q}(x | \alpha) = q^{-\alpha x} \sum_{k=0}^n \binom{n}{k} q^{\alpha k x} \tilde{E}_{k,q}(\alpha) [x : q^\alpha]^{n-k}.$$

Let  $\Gamma(s)$  be Euler-gamma function. Then the analytic continuation can be get as

$$\begin{aligned} n &\mapsto s \in \mathbb{R}, x \mapsto w \in \mathbb{C}, \\ \tilde{E}_{n,q}(\alpha) &\mapsto \tilde{E}_q(k + s - [s] : \alpha) = \tilde{\zeta}_{E,q}(-(k + s - [s]) | \alpha), \\ \binom{n}{k} &= \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} \mapsto \frac{\Gamma(s+1)}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)} \\ \tilde{E}_{s,q}(w | \alpha) &\mapsto \tilde{E}_q(s, w : \alpha) = q^{-\alpha w} \sum_{k=-1}^{[s]} \frac{\Gamma(s+1) \tilde{E}_q(k + (s - [s]) : \alpha) q^{\alpha w(k + (s - [s]))}}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)} [w : q^\alpha]^{[s]-k} \\ &= q^{-\alpha w} \sum_{k=0}^{[s]+1} \frac{\Gamma(s+1) \tilde{E}_q(-1+k+(s-[s]) : \alpha) q^{\alpha w(k-1+(s-[s]))}}{\Gamma(k+(s-[s]))\Gamma(2+[s]-k)} [w : q^\alpha]^{[s]+1-k}. \end{aligned}$$

Here  $[s]$  gives the integer part of  $s$ , and so  $s - [s]$  gives the fractional part.

Deformation of the curve  $\widetilde{E}_q(1, w : \alpha)$  into the curve of  $\widetilde{E}_q(2, w : \alpha)$  is by means of the real analytic cotinuation  $\widetilde{E}_q(s, w : \alpha)$ ,  $1 \leq s \leq 2$ ,  $-0.5 \leq w \leq 0.5$ .

**Acknowledgement 1.** *The third author would like to thank the Association SARA-GHU à Marseille for their hospitality during his stay there, when the work for this paper was done and dedicated this paper to Neda Agha-Soltan.*

## REFERENCES

- [1] T. Kim, Analytic continuation of  $q$ -Euler numbers and polynomials, Applied Mathematics Letters 21 (2008) 1320-1323.
- [2] T. Kim, On explicit formulas of  $p$ -adic  $q$ - $L$ -functions, Kyushu J. Math. 43 (1994) 73-86.
- [3] T. Kim, On  $p$ -adic interpolating function for  $q$ -Euler numbers and its derivatives, J. Math. Anal. Appl. 339 (2008) 598-608.
- [4] T. Kim, On the  $q$ -extension of Euler and Genocchi numbers, J. Math. Anal. Appl. 326 (2007) 1458-1465.
- [5] T. Kim, On a  $q$ -analogue of the  $p$ -adic log gamma functions and related integrals, Journal of Number Theory 76 (1999) 320-329.
- [6] T. Kim, On the analogs of Euler numbers and polynomials associated with  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  at  $q = -1$ , J. Math. Anal. Appl. 331 (2007) 779-792.
- [7] T. Kim, A. Bayad, Y-H. Kim, A study on the  $p$ -adic  $q$ -integrals representation on  $\mathbb{Z}_p$  associated with the weighted  $q$ -Bernstein and  $q$ -Bernoulli polynomials, Journal of Inequalities and Applications, Article ID 513821, 8 pages, doi:10.1155/2011/513821.
- [8] T. Kim, On the weighted  $q$ -Bernoulli numbers and polynomials, Advanced Studies in Contemporary Mathematics 21 (2011), no. 2, p. 207-215.
- [9] T. Kim,  $q$ -Volkenborn integration, Russ. J. Math. phys. 9 (2002), page 288-299.
- [10] T. Kim, An invariant  $p$ -adic  $q$ -integrals on  $\mathbb{Z}_p$ , Applied Mathematics Letters, vol. 21, pp. 105-108, 2008.
- [11] T. Kim,  $q$ -Euler numbers and polynomials associated with  $p$ -adic  $q$ -integrals, J. Nonlinear Math. Phys., 14 (2007), No. 1, 15-27.
- [12] T. Kim, Some identities on the  $q$ -Euler polynomials of higher order and  $q$ -Stirling numbers by the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$ , Russ. J. Math. Phys., 16 (2009), No.4, 484-491.
- [13] T. Kim, Analytic continuation of multiple  $q$ -zeta functions and their values at negative integers, Russ. J. Math. Phys. 11 (1) (2004) 71-76.
- [14] C. S. Ryoo, A note on the weighted  $q$ -Euler numbers and polynomials, Adv. Stud. Contemp Math. 21 (2011), 47-54.
- [15] C. S. Ryoo and T. Kim, An analogue of the zeta function and its applications, Applied Mathematics Letters 19 (2006), 1068-1072.
- [16] C. S. Ryoo, Some relations between  $q$ -Euler numbers and polynomials with weight  $(\alpha, \beta)$  and  $q$ -Bernstein polynomials with weight  $\alpha$ , Applied Mathematical Sciences, Vol. 6, no. 45, 2227-2234.
- [17] Y. Simsek, Twisted  $(h, q)$ -Bernoulli numbers and polynomials related to twisted  $(h, q)$ -zeta function and  $L$ -function, J. Math. Anal. Appl., 324(2006), 790-804.
- [18] Y. Simsek, On  $p$ -adic twisted  $q$ - $L$ -functions related to generalized twisted Bernoulli numbers, Russian J. Math. Phys., 13(3) (2006), 340-348.
- [19] K. Hensel, Theorie der Algebraischen Zahlen I. Teubner, Leipzig, 1908.
- [20] L. Carlitz,  $q$ -Bernoulli Numbers and Polynomials, Duke Math. J. 15, 987-1000 (1948).
- [21] L. Carlitz,  $q$ -Bernoulli and Eulerian Numbers, Trans. Amer. Math. Soc. 76, 332-350 (1954).
- [22] J. J. Seo and S. Araci, Some Properties involving the higher order  $q$ -Genocchi numbers and polynomials with weight  $(\alpha, \beta)$  via the  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$ , Journal of Chungcheon Mathematical Society, Vol 24, No. 4, December 2011.
- [23] H. Jolany, S. Araci, M. Acikgoz and J. J. Seo, A note on the generalized  $q$ -Genocchi measure with weight  $\alpha$ , Bol. Soc. Paran. Math. v. 31 1 (2013): pp. 17-27 (in press).
- [24] S. Araci, D. Erdal, J. J. Seo, A study on the fermionic  $p$ -adic  $q$ -integral representation on  $\mathbb{Z}_p$  associated with weighted  $q$ -Bernstein and  $q$ -Genocchi polynomials, Abstract and Applied Analysis, Volume 2011, Article ID 649248, 10 pages.

- [25] S. Araci, M. Acikgoz and J. J. Seo, A study on the weighted  $q$ -Genocchi numbers and polynomials with their interpolation function, Honam Mathematical J. 34 (2012), No. 1, pp. 11-18.
- [26] S. Araci, M. Acikgoz and A. Gürsul, Analytic continuation of weighted  $q$ -Genocchi numbers and polynomials, <http://arxiv.org/pdf/1204.1996v2.pdf>.
- [27] S. Araci, M. Acikgoz and K. H. Park, A note on the  $q$ -analogue of Kim's  $p$ -adic log gamma type functions associated with  $q$ -extension of Genocchi and Euler numbers with weight  $\alpha$ , accepted in Bulletin of the Korean Mathematical Society.
- [28] S. Araci, M. Acikgoz, K. H. Park and H. Jolany, On the unification of two families of multiple twisted type polynomials by using  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  at  $q = 1$ , accepted in Bulletin of the Malaysian Mathematical Sciences and Society.
- [29] S. Araci, J. J. Seo and D. Erdal, New construction weighted  $(h, q)$ -Genocchi numbers and polynomials related to Zeta type function, Discrete Dynamics in Nature and Society, Volume 2011, Article ID 487490, 7 pages, doi:10.1155/2011/487490.
- [30] S. Araci, N. Aslan and J. J. Seo, A Note on the weighted twisted Dirichlet's type  $q$ -Euler numbers and polynomials, Honam Mathematical J. 33 (2011), no. 3, pp. 311-320.

**University of Gaziantep, Faculty of Science and Arts, Department of Mathematics,  
27310 Gaziantep, TURKEY**

*E-mail address:* mtsrkn@hotmail.com.tr

**University of Gaziantep, Faculty of Science and Arts, Department of Mathematics,  
27310 Gaziantep, TURKEY**

*E-mail address:* acikgoz@gantep.edu.tr

**School of Mathematics, Statistics and Computer Science, University of Tehran,  
Iran**

*E-mail address:* hassan.jolany@khayam.ut.ac.ir